Pro-unipotent completion

by Victor

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k field.

I Pro

Def. [Filtered cat] $\mathcal{D}$ small and 1) $\forall x, y \in \mathcal{D}\ \exists z$ $\begin{matrix} x \searrow \\ z \\ y \nearrow \end{matrix}$

2) $a \underset{g}{\overset{f}{\rightrightarrows}} b\ \exists\, b \xrightarrow{h} c$ s.t. $hf = hg$

[Cofiltered = dual]

Def [Ind objects] $\mathcal{C}$ cat.

Objects: $F : \underset{\substack{\uparrow \\ \text{filtered}}}{\mathcal{D}} \longrightarrow \mathcal{C}$ Morphisms: $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{E} \to \mathcal{C}$

$$\mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(F, G) = \lim_{d \in \mathcal{D}} \operatorname{colim}_{e \in \mathcal{E}} \mathrm{Hom}_{\mathcal{C}}(F(d), G(e))$$

i.e. $\operatorname{colim} \mathcal{D}^{op} \times \mathcal{E} \xrightarrow{F^{op} \times G} \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,-)} \mathrm{Set}$

[Pro objects]

Objects: $F : \underset{\substack{\uparrow \\ \text{cofiltered}}}{\mathcal{D}} \longrightarrow \mathcal{C}$ Morphisms: $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{E} \to \mathcal{C}$

$$\mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}}(F, G) = \lim_{e \in \mathcal{E}} \operatorname{colim}_{d \in \mathcal{D}} \mathrm{Hom}_{\mathcal{C}}(F(d), G(e))$$

i.e. $\lim \mathcal{D}^{op} \times \mathcal{E} \xrightarrow{F^{op} \times G} \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,-)} \mathrm{Set}$

Prop. 1) $\mathcal{C} \simeq \mathcal{C}'$ then $\mathrm{Ind}(\mathcal{C}) \simeq \mathrm{Ind}(\mathcal{C}')$ and $\mathrm{Pro}(\mathcal{C}) \simeq \mathrm{Pro}(\mathcal{C}')$

2) $\mathcal{C} \simeq (\mathcal{C}')^{op}$ then $(\mathrm{Ind}\,\mathcal{C})^{op} \simeq \mathrm{Pro}(\mathcal{C}')$ and $(\mathrm{Pro}\,\mathcal{C})^{op} \simeq \mathrm{Ind}(\mathcal{C}')$

Prop. $\mathcal{C}' \hookrightarrow \mathcal{C}$ subcat.

1) $\mathcal{C}$ cocomplete

2) $\exists\, \delta : \mathcal{C} \longrightarrow \mathrm{Ind}(\mathcal{C}')$ s.t.
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\delta"] \arrow[rr, bend left, "\mathrm{Id}_{\mathcal{C}}"] & \mathrm{Ind}\,\mathcal{C}' \arrow[r, "\operatorname{colim}"] & \mathcal{C} \end{tikzcd}$$

3) Every $c \in \mathcal{C}'$ is compact in $\mathcal{C}$ (i.e. $\mathrm{Hom}(c, -) : \mathcal{C} \longrightarrow \mathrm{Set}$)

Then: $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}')$

Examples. 1) Vector space: $\mathrm{Vect}_k \cong \mathrm{Ind}(\mathrm{Vect}_k^{f.d})$

2) Grp is complete:

$$\mathrm{Pro}(\mathrm{Grp}^{\text{finite}}) \xrightarrow{\lim} \mathrm{Grp}$$

If $G \in \mathrm{Grp}$ then $\delta: \mathrm{Grp} \longrightarrow \mathrm{Pro}(\mathrm{Grp}^{\text{finite}})$

$$G \longmapsto \{G/N : N \trianglelefteq_{\text{finite}} G\} \xmapsto{\lim} \hat{G} := \lim G/N$$

3) [Sweedler] $\mathrm{Coalg} \simeq \mathrm{Ind}(\mathrm{Coalg}^{f.d})$

We have $\mathrm{Alg}^{f.d} \simeq \mathrm{Coalg}^{f.d} \Rightarrow \mathrm{Pro}(\mathrm{Alg}^{f.d}) \xrightarrow{\sim} \mathrm{Coalg}$.

4) Hopf alg.
$$\left\{\begin{matrix}\text{affine}\\ \text{algebraic}\\ \text{groups}\end{matrix}\right\} \underset{\mathrm{Spec}(-)}{\overset{\mathcal{O}_-}{\rightleftarrows}} \left\{\begin{matrix}\text{fin. presented}\\ \text{commutative}\\ \text{Hopf alg.}\end{matrix}\right\}$$

Because $\mathrm{Spec}(A \otimes_k B) \cong \mathrm{Spec}(A) \times_{\mathrm{Spec}\,k} \mathrm{Spec}(B)$

$$\left\{\begin{matrix}\text{comm.}\\ \text{Hopf}\\ \text{algebras}\end{matrix}\right\} \cong \mathrm{Ind}\left\{\begin{matrix}\text{fin. presented}\\ \text{commutative}\\ \text{Hopf alg.}\end{matrix}\right\} \Longrightarrow \mathrm{Pro}\left(\begin{matrix}\text{affine}\\ \text{algebraic}\\ \text{groups}\end{matrix}\right)^{op} \simeq \begin{matrix}\text{comm.}\\ \text{Hopf}\\ \text{algebras}\end{matrix}$$

II Unipotent

Def. $G \hookrightarrow GL_n/g^n=1$

$\Leftrightarrow$ G is isomorphic to a subgroup of $\mathbb{U}T_n = \left\{\begin{pmatrix}1 & * & * \\ & \ddots & \\ 0 & & 1\end{pmatrix} \in GL_n\right\}$

ex. $G = G_0 \supseteq \cdots \supseteq G_n = \{e\}$ with $G_i/G_{i+1} \cong \mathbb{G}_a$ additive

k is perfect. $\Rightarrow$ $G(\mathbb{Q}) = G_0(\mathbb{Q}) \supseteq \cdots \supseteq \{e\}$ with $G_i/G_{i+1}(\mathbb{Q}) \cong \mathbb{Q}$

Thm. $\{G \text{ is unipotent}\} \Leftrightarrow \{\mathcal{O}_G \text{ is a conilpotent coalg.}\}$

Sketch of pf.

G unipotent

$\Leftrightarrow$ $\forall$ rep V of G $\exists v \in V$ $G \cdot v = v$

$\Leftrightarrow$ $\forall \mathcal{O}_G$-comodule $\exists$ an exhaustive filtration

$\Leftrightarrow$ $\mathcal{O}_G$ conilpotent. □

Def. Coalg. $\mathcal{C}$ is conilpotent if the filtration
$$F_n\mathcal{C} = \{x \in \mathcal{C} \mid \bar{\Delta}^n(x) = 0\} \subseteq F_{n+1}\mathcal{C}$$
is exhaustive.

$$\bar{\Delta} := \Delta - 1 \otimes \mathrm{Id} - \mathrm{Id} \otimes 1.$$

i.e.
$$\mathcal{C} \cong \operatorname{colim}_n F_n\mathcal{C}$$

Conclusion. $\mathrm{Pro}\begin{pmatrix}\text{unipotent affine}\\ \text{algebraic group}\end{pmatrix} \cong \begin{matrix}\text{comm. conilpotent}\\ \text{Hopf algebras}\end{matrix}$

III Completions.

G an abstract group. How to get a prounipotent aff. alg gp?

take $k[G] = \bigoplus_{g \in G} kg$ with $\mu : k[G] \otimes k[G] \longrightarrow k[G] \quad g \otimes g' \longmapsto gg'$

$\Delta : k[G] \longrightarrow k[G] \otimes k[G] \quad g \longmapsto g \otimes g$

Fact: $k[G]$ cocomm. Hopf algebra.

Def. A an algebra is complete if the filtration

$F_n A := \{ x \in A \mid$ x can be written as a product of n non-unit elts $\}$

is complete i.e. $A \cong \lim A/F_n A$

Def. Complete algebra A is pro-nilpotent if A/F_nA fin.dim.

$$\begin{tikzcd}
\mathrm{Pro}(\mathrm{nil\,Alg}^{f.d.}) & \subseteq & \mathrm{Pro}(\mathrm{Alg}^{f.d}) \arrow[r, "*"] \arrow[dd, "\lim"] & \mathrm{Ind}(\mathrm{conil\,Coalg}^{f.d}) & \subseteq & \mathrm{Ind}(\mathrm{Coalg}^{f.d}) \arrow[dd, "\mathrm{colim}"] \\
\{A/F_nA\} \arrow[rrr, mapsto] \arrow[d, mapsto, "\lim"'] & & & \{(A/F_nA)^*\} \arrow[d, mapsto, "\mathrm{colim}"] & & \\
A \in \text{pronilpotent} & & \mathrm{Alg} & A^\vee \in \mathrm{conil\,Coalg} & \subseteq & \mathrm{Coalg}
\end{tikzcd}$$

Fact: This extends to Hopf algebras:

Prop. F_1A/F_2A fin.dim $\iff$ A/F_nA fin.dim.

proof. Enough to show $\left(F_1A/F_2A\right)^{\otimes n} \twoheadrightarrow F_nA/F_{n+1}A$ ... $\square$

Prop. Let $k[G] \xrightarrow{\varepsilon} k$ and $I := \ker(\varepsilon)$.

1) $F_n(k[G]) = I^n$

2) $I/I^2 \cong G^{ab} \otimes_{\mathbb{Z}} k$

Conclusion.

Define: $k[G]^\wedge_I := \lim k[G]/I^n$

then:

$k[G]^\wedge_I$ pronilpotent $\iff$ $G^{ab} \otimes_{\mathbb{Z}} k$ fin.dim.

Define:

$G^{uni} := \mathrm{Spec}\left((k[G]^\wedge_I)^\vee \right)$

Thm.

1) G^{uni} is pro-unipotent.

2) Let Γ be a pro-unipotent group then

$$\begin{tikzcd} G \arrow[r] \arrow[d, "f"'] & G^{uni}(k) \arrow[dl, "\exists! \tilde{f}"] \\ \Gamma^{uni}(k) & \end{tikzcd}$$

Note: $G^{uni}(k) = \mathrm{Grplike}(k[G]^{\wedge}_{I})$

proof.

$G^{uni}(k) := \mathrm{Hom}(\mathrm{Spec}(k), G^{uni})$

$\cong \mathrm{Hom}_{k\text{-alg}}((k[G]^{\wedge}_{I})^{\vee}, k)$

$\cong \mathrm{Grplike}(k[G]^{\wedge}_{I})$. $\square$

Let $\mathrm{char}(k) = 0$.

$\mathfrak{g}$ Lie alg $\rightsquigarrow$ $\mathbb{U}\mathfrak{g}$ cocomm Hopf alg

$\rightsquigarrow (\mathbb{U}\mathfrak{g})^{\wedge}_{I}$ pronilpotent

so need $I\mathbb{U}\mathfrak{g} / I^2\mathbb{U}\mathfrak{g} \cong \mathfrak{g}^{ab}$ fin. dim.

$\mathfrak{g}^{ab} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$

$((\mathbb{U}\mathfrak{g})^{\wedge}_{I})^{\vee}$ conilpotent

$\rightsquigarrow G^{uni}_{\mathfrak{g}} \xleftrightarrow{\text{Lie}} \mathfrak{g}^{uni} := \mathrm{prim}\,((\mathbb{U}\mathfrak{g})^{\wedge}_{I})^{\vee}$

Fact: $\mathrm{prim}\,(\mathbb{U}\mathfrak{g})^{\wedge}_{I} \cong I(\mathbb{U}\mathfrak{g}^{\wedge}_{I})^{\vee} / I^2 \cong \mathrm{Lie}(G^{uni}_{\mathfrak{g}})$

IV Examples

1) $G = \mathbb{Z}$ then
$\mathbb{Z}^{uni}(k) = k$ $\quad$ $\mathrm{char}\, k = 0$
$\mathbb{Z}^{uni}(k) = \mathbb{Z}_p$ $\quad$ $\mathrm{char}\, k = p$

2) $G = \mathbb{Z}/n\mathbb{Z}$ then
$\mathbb{Z}/n\mathbb{Z}^{uni}(k) = 0$ $\quad$ $\mathrm{char}\, k = 0$ or $\mathrm{char}(k) \nmid n$
$\mathbb{Z}/n\mathbb{Z}^{uni}(k) = \mathbb{Z}/p^d\mathbb{Z}$ $\quad$ if $n = p^d \cdot r$ and $\mathrm{char}(k) = p$

3) G abelian then
$G^{uni}(k) = G \otimes_{\mathbb{Z}} k$ $\quad$ $\mathrm{char}\, k = 0$
$G^{uni}(k) = G \otimes_{\mathbb{Z}} \mathbb{Z}_p$ $\quad$ $\mathrm{char}\, k = p$

4) PaB:
A) It extends to groupoids.
B) $PB_n^{ab} \cong \mathbb{Z}^{\frac{n(n-1)}{2}}$ $\Rightarrow$ fin. dim.
C) The completion is a monoidal functor.

~ Summary ~

$$
\begin{tikzcd}
(\mathrm{Grp}, \times) \arrow[r, "k{[-]}"] & (\mathrm{Hopf}^{\mathrm{com}}\text{-}\mathrm{alg}, \otimes) \arrow[r, "(-)^{\wedge}_{I}"] & (\mathrm{pro}(\mathrm{Hopf}^{\mathrm{com}}\text{-}\mathrm{alg})_{\text{nilpotent}}, \otimes) \arrow[dl, "\mathrm{Grplike}"'] \arrow[dd, "(-)^{\vee}"] \\
& (\mathrm{pro}(\mathrm{Grp}^{\mathrm{uni}}), \times) \arrow[dr, phantom, "\circlearrowright"] & \\
& (\mathrm{pro}\ \text{unipotent alg. grp}, \times_{\mathrm{Sp}}) \arrow[u, "(-)(k)"] & (\mathrm{pro}(\mathrm{Hopf}^{\mathrm{com}}\text{-}\mathrm{alg})_{\text{conilpotent}}, \otimes) \arrow[l, "\mathrm{Spec}(-)"']
\end{tikzcd}
$$