

Pro-unipotent completion

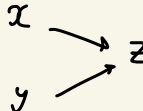
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Oct 15 2020

I Pro II Unipotent III Completions IV Examples.

k field.

I Pro

Def. [Filtered cat] \mathcal{D} small and 1) $\forall x, y \in \mathcal{D} \exists z$ 

[Cofiltered = dual]

2) $a \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} b \exists b \xrightarrow{h} c$ s.t. $hf = hg$

Def [Ind objects] \mathcal{C} cat.

Objects: $F: \mathcal{D} \rightarrow \mathcal{C}$
 \uparrow
filtered

Morphisms: $F: \mathcal{D} \rightarrow \mathcal{C}$
 $G: \mathcal{E} \rightarrow \mathcal{C}$

$$\text{Hom}_{\text{Ind } \mathcal{C}}(F, G) = \lim_{d \in \mathcal{D}} \text{colim}_{e \in \mathcal{E}} \text{Hom}_{\mathcal{C}}(F(d), G(e))$$

$$\text{i.e. } \text{colim } \mathcal{D}^{\text{op}} \times \mathcal{E} \xrightarrow{F^{\text{op}} \times G} \mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\text{Hom}_{\mathcal{C}}(-, -)} \text{Set}$$

[Pro objects]

Objects: $F: \mathcal{D} \rightarrow \mathcal{C}$
 \uparrow
cofiltered

Morphisms: $F: \mathcal{D} \rightarrow \mathcal{C}$
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$$\text{i.e. } \lim \mathcal{D}^{\text{op}} \times \mathcal{E} \xrightarrow{F^{\text{op}} \times G} \mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\text{Hom}_{\mathcal{C}}(-, -)} \text{Set}$$

Prop. 1) $\mathcal{C} \simeq \mathcal{C}'$ then $\text{Ind}(\mathcal{C}) \simeq \text{Ind}(\mathcal{C}')$ and $\text{Pro}(\mathcal{C}) \simeq \text{Pro}(\mathcal{C}')$

2) $\mathcal{C} \simeq (\mathcal{C}')^{\text{op}}$ then $(\text{Ind } \mathcal{C})^{\text{op}} \simeq \text{Pro}(\mathcal{C}')$ and $(\text{Pro } \mathcal{C})^{\text{op}} \simeq \text{Ind}(\mathcal{C}')$

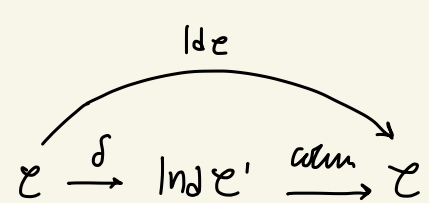
Prop. $\mathcal{C}' \hookrightarrow \mathcal{C}$ subcat.

1) \mathcal{C} cocomplete

2) $\exists \delta: \mathcal{C} \rightarrow \text{Ind}(\mathcal{C}')$ s.t. $\mathcal{C} \xrightarrow{\delta} \text{Ind } \mathcal{C}' \xrightarrow{\text{colim}} \mathcal{C}$

3) Every $c \in \mathcal{C}'$ is compact in \mathcal{C} (i.e. $\text{Hom}(c, -): \mathcal{C} \rightarrow \text{Set}$)

Then: $\mathcal{C} \simeq \text{Ind}(\mathcal{C}')$



Examples. 1) Vector space: $\text{Vect}_k \cong \text{Ind}(\text{Vect}_k^{\text{f.d.}})$

2) Grp is complete:

$$\text{Pro}(\text{Grp}^{\text{finite}}) \xrightarrow{\text{lim}} \text{Grp}$$

$$\text{If } G \in \text{Grp} \text{ then } \delta: \text{Grp} \longrightarrow \text{Pro}(\text{Grp}^{\text{finite}})$$

$$G \longmapsto \left\{ G/N : N \trianglelefteq G \right\}_{\text{finite}} \xrightarrow{\text{lim}} \hat{G} := \varprojlim G/N$$

3) [Sweedler] $\text{Coalg} \cong \text{Ind}(\text{Coalg}^{\text{f.d.}})$

$$\text{We have } \text{Alg}^{\text{f.d.}} \cong \text{Coalg}^{\text{f.d.}} \implies \text{Pro}(\text{Alg}^{\text{f.d.}}) \xrightarrow{\sim} \text{Coalg}.$$

$$4) \text{Hopt alg. } \left\{ \begin{array}{l} \text{affine} \\ \text{algebraic} \\ \text{groups} \end{array} \right\} \xrightleftharpoons[\text{Spec}(-1)]{\mathcal{O}_-} \left\{ \begin{array}{l} \text{fin. presented} \\ \text{commutative} \\ \text{Hopt alg.} \end{array} \right\}$$

$$\text{Because } \text{Spec}(A \otimes_k B) \cong \text{Spec}(A) \times_{\text{Spec} k} \text{Spec}(B)$$

$$\left\{ \begin{array}{l} \text{comm.} \\ \text{Hopt} \\ \text{algebras} \end{array} \right\} \cong \text{Ind} \left\{ \begin{array}{l} \text{fin. presented} \\ \text{commutative} \\ \text{Hopt alg.} \end{array} \right\} \implies \text{Pro}(\text{affine algebraic groups})^{\text{op}} \cong \text{comm. Hopt algebras}$$

II Unipotent

Def. $G \longleftrightarrow \text{GL}_n / \mathfrak{g}^n = 1$

$\iff G$ is isomorphic to a subgroup of

$$\text{UT}_n = \left\{ \begin{pmatrix} 1 & * & * \\ & \ddots & * \\ 0 & & 1 \end{pmatrix} \in \text{GL}_n \right\}$$

ex.

$$G = G_0 \supseteq \dots \supseteq G_n = \{e\} \text{ with } G_i/G_{i+1} \cong G_a \text{ additive}$$

k is perfect. \implies

$$G(\mathbb{Q}) = G_0(\mathbb{Q}) \supseteq \dots \supseteq \{e\} \text{ with } G_i/G_{i+1}(\mathbb{Q}) \cong \mathbb{Q}$$

$$\text{Thm. } \left\{ G \text{ is unipotent} \right\} \iff \left\{ \mathcal{O}_G \text{ is a conilpotent coalg.} \right\}$$

sketch of pf.

G unipotent

$$\iff \forall \text{ rep } V \text{ of } G \exists v \in V \text{ } G \cdot v = v$$

$$\iff \forall \mathcal{O}_G\text{-comodule } \exists \text{ an exhaustive filtration}$$

$$\iff \mathcal{O}_G \text{ conilpotent. } \square$$

Def. Coalg. \mathcal{C} is conilpotent if the filtration $F_n \mathcal{C} = \{x \in \mathcal{C} \mid \Delta^n(x) = 0\} \subseteq F_{n+1} \mathcal{C}$ is exhaustive.

$$\Delta := \Delta - 1 \otimes 1 - 1 \otimes 1.$$

i.e.

$$\mathcal{C} \cong \varprojlim_n F_n \mathcal{C}$$

Conclusion.

$$\text{Pro}(\text{unipotent affine algebraic group}) \cong \text{comm. conilpotent Hopt algebras}$$

III Completions.

G an abstract group. How to get a pronilpotent aff. alg \mathfrak{g} ?

take $k[G] = \bigoplus_{g \in G} kg$ with $M: k[G] \otimes k[G] \rightarrow k[G] \quad g \otimes g' \mapsto gg'$
 $\Delta: k[G] \rightarrow k[G] \otimes k[G] \quad g \mapsto g \otimes g$

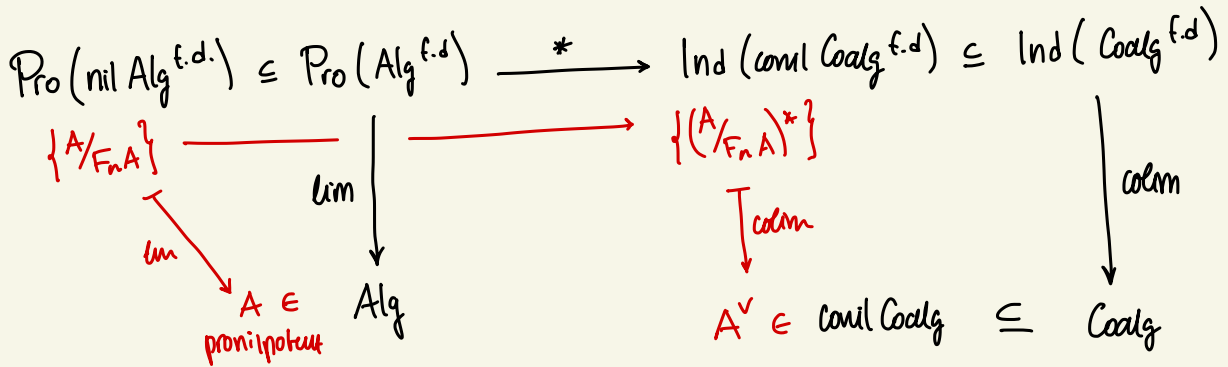
Fact: $k[G]$ cocomm. Hopf algebra.

Def. An algebra is complete if the filtration

$$F_n A := \{x \in A \mid x \text{ can be written as a product of } n \text{ non-unit elts}\}$$

is complete i.e. $A \cong \varprojlim A/F_n A$

Def. Complete algebra A is pronilpotent if $A/F_n A$ fm.d.m.



Fact: This extends to Hopf algebras:

Prop. $F_1 A / F_2 A$ fm.d.m. $\iff A / F_n A$ fm.d.m.

proof. Enough to show $(F_1 A / F_2 A)^{\otimes n} \longrightarrow F_n A / F_{n+1} A \dots \square$

Prop. Let $k[G] \xrightarrow{\varepsilon} k$ and $I := \ker(\varepsilon)$.

- 1) $F_n(k[G]) = I^n$
- 2) $I/I^2 \cong G^{ab} \otimes_{\mathbb{Z}} k$

Conclusion.

Define: $k[G]_{\mathbb{I}}^{\wedge} := \varprojlim k[G]/I^n$

then:

$k[G]_{\mathbb{I}}^{\wedge}$ pronilpotent $\iff G^{ab} \otimes_{\mathbb{Z}} k$ fm.d.m.

Define:

$$G^{\text{uni}} := \text{Spec}((k[G]_{\mathbb{I}}^{\wedge})^{\vee})$$

Thm.

- 1) G^{uni} is pro-unipotent.
- 2) Let Γ be a pro-unipotent group

then

$$\begin{array}{ccc}
 G & \longrightarrow & G^{\text{uni}}(k) \\
 \downarrow f & & \swarrow \exists! \tilde{f} \\
 \Gamma^{\text{uni}}(k) & &
 \end{array}$$

Note: $G^{\text{uni}}(k) = \text{Grp}_{\text{line}}(k[G]_{\pm}^{\wedge})$

proof.

$$\begin{aligned}
 G^{\text{uni}}(k) &:= \text{Hom}(\text{Spec}(k), G^{\text{uni}}) \\
 &\cong \text{Hom}_{k\text{-alg}}((k[G]_{\pm}^{\wedge})^{\vee}, k) \\
 &\cong \text{Grp}_{\text{line}}(k[G]_{\pm}^{\wedge}). \quad \square
 \end{aligned}$$

Let $\text{char}(k) = 0$.

\mathfrak{g} Lie alg $\rightsquigarrow \mathcal{U}\mathfrak{g}$ cocomm Hopf alg

$\rightsquigarrow (\mathcal{U}\mathfrak{g})_{\pm}^{\wedge}$ pronilpotent

so need $\mathcal{U}\mathfrak{g}/\mathcal{I}^2\mathcal{U}\mathfrak{g} \cong \mathfrak{g}^{\text{ab}}$ fin. dim.
 $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$

$((\mathcal{U}\mathfrak{g})_{\pm}^{\wedge})^{\vee}$ conilpotent

$\rightsquigarrow G_{\mathfrak{g}}^{\text{uni}}$

$\xrightarrow{\text{Lie}}$

$\mathfrak{g}^{\text{uni}} := \text{prim}((\mathcal{U}\mathfrak{g})_{\pm}^{\wedge})^{\vee}$

Fact: $\text{prim}((\mathcal{U}\mathfrak{g})_{\pm}^{\wedge}) \cong \mathcal{I}((\mathcal{U}\mathfrak{g})_{\pm}^{\wedge})^{\vee} / \mathcal{I}^2 \cong \text{Zie}(G_{\mathfrak{g}}^{\text{uni}})$

IV Examples

1) $G = \mathbb{Z}$ then $\mathbb{Z}^{\text{uni}}(k) = k$ $\text{char } k = 0$
 $\mathbb{Z}^{\text{uni}}(k) = \mathbb{Z}_p$ $\text{char } k = p$

2) $G = \mathbb{Z}/n\mathbb{Z}$ then $\mathbb{Z}/n\mathbb{Z}^{\text{uni}}(k) = 0$ $\text{char } k = 0$ or $\text{char}(k) \nmid n$
 $\mathbb{Z}/n\mathbb{Z}^{\text{uni}}(k) = \mathbb{Z}/p^d\mathbb{Z}$ if $n = p^d \cdot r$ and $\text{char}(k) = p$

3) G abelian then $G^{\text{uni}}(k) = G \otimes_{\mathbb{Z}} k$ $\text{char } k = 0$
 $G^{\text{uni}}(k) = G \otimes_{\mathbb{Z}} \mathbb{Z}_p$ $\text{char } k = p$

- 4) PB:
- A) It extends to groupoids.
 - B) $\text{PB}_n^{\text{ab}} \cong \mathbb{Z}^{\frac{n(n-1)}{2}}$ \Rightarrow fin. dim.
 - C) The completion is a monoidal functor.

~ Summary ~

